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A MODEL COMPARISON IN LEAST SQUARES COLLOCATION. (U)
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UNCLASSIFIED DGS-238 AFGL-TR-76-0198 NL

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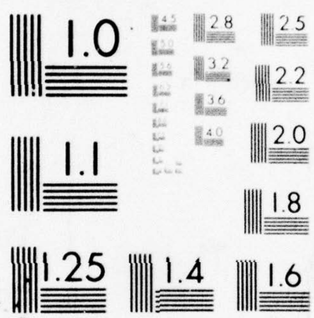

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Reports of the Department of Geodetic Science, No. 238

A MODEL COMPARISON IN LEAST SQUARES COLLOCATION

Reiner Rummel

The Ohio State University
Research Foundation
Columbus, Ohio 43212

August, 1976

Scientific Report No. 5

AIR FORCE GEOPHYSICS LABORATORY
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
HANSCOM AFB, MASSACHUSETTS 01731

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 18 AFGL-TR-76-0198
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER: 9 Interim rept.,
4. TITLE (and Subtitle): 6 A MODEL COMPARISON IN LEAST SQUARES COLLOCATION,
5. TYPE OF REPORT & PERIOD COVERED: Scientific Interim Scientific ~~Report No.~~ 5
6. PERFORMING ORG. REPORT NUMBER: Dept. of Geodetic Science No. 238
7. AUTHOR(s): 10 Reiner Rummel
8. CONTRACT OR GRANT NUMBER(s): 15 F19628-76-C-0010
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Geodetic Science 400254 The Ohio State University ~~Research Foundation~~ 1314 Kinnear Road, Columbus, Ohio 43212
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 62101F 76003302 16 17 03
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Contract Monitor: Bela Szabo/ (LW)
12. REPORT DATE: 11 Aug[redacted]76
13. NUMBER OF PAGES: 44
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 44p.
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

14 DGS-238, Scientific-5

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Physical geodesy, collocation, least squares estimation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

→ In the model equation of least squares collocation there appears a random signal, s', which is of the same physical nature as the observations whereas the signal to be estimated, s, is only linked to the model through its correlation with s'. This Model (1) is compared with the least squares estimation Model (2), where the signal to be estimated is directly part of → next page

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cont

the model with a coefficient matrix R. The basic differences of these two models in the framework of physical geodesy are pointed out by analyzing the validity of the equation

$$s' = R\,s,$$

that transforms one model into the other, for different cases. For clarification purposes least squares filtering, prediction and collocation are discussed separately. In filtering problems the coefficient matrix R becomes the unit matrix and by this the two models become identical. For prediction and collocation problems the relation $s' = R\,s$ is only fulfilled in the global limit where s becomes either a continuous function on the earth or an infinite set of spherical harmonic coefficients. Applying Model (2), we see that for any finite dimension of s the operator equations of physical geodesy are approximated by a finite matrix relation whereas in Model (1) the operator equations are applied in their correct form on a continuous, approximate functions $\tilde{s}$.

Both methods have been applied in a numerical example where spherical harmonic coefficients of the geoid height were estimated from geoid heights given in a global regular point grid over the sphere. The results show that not only the specific features of the two least squares estimation methods have to be taken into consideration but to a high extent also the characteristics of the involved gravity quantities when a decision has to be made which of both methods should be applied.

Foreword

This report was prepared by Dr. Reiner Rummel, Visiting Research Associate, Department of Geodetic Science, The Ohio State University, under Air Force Contract No. F19628-76-C-0010, The Ohio State University Research Foundation Project No. 4214B1, Project Supervisor, Richard H. Rapp, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Geophysics Laboratory (AFGL), Hanscom Air Force Base, Massachusetts, with Mr. Bela Szablo, Project Scientist.

Acknowledgement

Valuable suggestions by Mr. T. Krarup, Dr. H. Moritz and Dr. R. H. Rapp are gratefully acknowledged. Some aspects of this paper arose in a joint discussion with Dr. D. Lelgemann, Dr. H. Moritz, Dr. R. H. Rapp, Dr. K. P. Schwarz, and Dr. U. A. Uotila.

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Part I: Theory

1. Introduction

The method of least squares collocation was introduced into physical geodesy in two principal presentations by T. Krarup (1969) and H. Moritz (1972). Meanwhile, the method -- in this context denoted as "Method One" -- is well established for different types of approximation and adjustment problems. Three features for application in physical geodesy are pointed out:

--- We assume a certain number of measurements of a function related to the anomalous potential of the earth's gravity field is given, such as gravity anomalies, surface densities, deflections of the vertical and so on. Then the use of a harmonic and regular covariance expression in collocation will lead to a global approximation to this function with minimum norm. The norm is defined by the covariance function.

--- The subsequent application of "the law of propagation of covariances" provides a global approximation to any desired function related to the earth's anomalous potential. In addition, due to the same fact, least squares collocation allows the combination of all types of measurements related to the anomalous potential.

--- A basic quality of exact collocation is the reproduction of the given data by the approximating function. This property is fulfilled in least squares collocation too, where the signal part of the measurements is reproduced by the approximating function.

Unfortunately, there are some numerical problems in contrast to the theoretical elegance of this method. Especially the necessary solution of a large linear system with a dimension equal to the number of observations can cause severe difficulties.

A similar least squares estimation method, in this context denoted as "Method Two", is described in (P. Whittle, 1963). A treatment of the method in geodesy, for example, is given by S. Lauer (1971), and by B. D. Tapley and B. E. Schutz (1973). Also Method Two is called least squares collocation in the geodetic literature, compare (H. Moritz, 1973, p. 84), (H. Moritz and K. P. Schwarz, 1973), and (K. P. Schwarz, 1974).

A generalisation in combining both types of least squares methods is derived by H. Wolf (1974).

K. P. Schwarz (1974) proved that for Method Two the solution of a linear system with a dimension equal to the number of observations can be replaced by a solution of a system with a dimension equal to the number of unknowns. This property is of great advantage for all overdetermined problems.

The purpose of the present paper is a comparison of the models underlying these two least squares estimation methods and an analysis of their individual features, since in the geodetic literature both methods are not clearly discriminated. In addition, we expect from the comparison, an answer to the question; if the solution of a large linear system in Method One, with a dimension equal to the number of observations, can be avoided as in Method Two. The considerations will be restricted to the application of these models to quantities related to the anomalous potential of the earth.

In a recent paper B. D. Tapley (1975) analyzes the two methods in relation to the minimum variance principle.

2. Basic Equations for the Two Models

A detailed exposition of the models and their least squares solution is given in the above mentioned literature. We restrict ourselves on the presentation of the basic formulas; for Method One, equations (1) to (6); the corresponding equations for Method Two are equations (7) to (12).

The model equation for Method One is

$$\underset{n1}{\ell} = \underset{nm}{A}\,\underset{m1}{x} + \underset{n1}{s'} + \underset{n1}{n} \tag{1}$$

ℓ ..... vector of observations, dim (n x 1),

x ..... unknown parameters, dim (m x 1),

A ..... coefficient matrix, dim (n x m), that relates the observations ℓ to the parameters x,

s' ..... random signal part of ℓ, dim (n x 1), with $E(s') = 0$ and $E(s's'^T) = C_{s's'}$, where $E(.)$ means the statistical expectation and is usually defined in physical geodesy to be the integral over the earth, $C_{s's'}$ is the covariance matrix for the signal s'.

n ..... random signal, with $E(n) = 0$ and $E(nn^T) = C_{nn}$. We also assume that $E(s'n^T) = 0$.

The desired signal is s with dimension (k x 1). The signal s does not appear explicitly in equation (1), it is linked to the model by a zero matrix 0. We obtain

$$\underset{n1}{\ell} = \underset{nm}{A}\,\underset{m1}{x} + \underset{n\;n+k}{\begin{bmatrix} 0 \\ I \end{bmatrix}^T} \underset{n+k\;1}{\begin{bmatrix} s \\ s'+n \end{bmatrix}} \tag{2}$$

or with

$$\begin{bmatrix} O \\ I \end{bmatrix} = B \text{ and } \begin{bmatrix} s \\ s' + n \end{bmatrix} = v \; ,$$

$$\ell = Ax + B^T v$$

where I ... identity matrix.

The minimum length of the vector v will be derived from the minimization of

$$v^T Q^{-1} v$$

where

(3) $$Q = \begin{bmatrix} C_{ss} & C_{ss'} \\ C_{s's} & C_{s's'} + C_{nn} \end{bmatrix}$$

and C_{ss} ..... covariance vector of s
$C_{ss'}$ .... crosscovariance between s and s'.

It is assumed that $E(sn^T) = 0$. From the least squares solution we obtain

$$v = QB^T (BQB^T)^{-1} (\ell - Ax)$$

and for the parameters x,

(4) $$x = (A^T (BQB^T)^{-1} A)^{-1} A^T (BQB^T)^{-1} \ell ,$$

where with equation (3)

$$Q B^{T} = \begin{bmatrix} C_{ss'} \\ C_{s's'} + C_{nn} \end{bmatrix} \quad \text{and} \quad B Q B^{T} = C_{s's'} + C_{nn}.$$

Thus, we obtain for the components of v

$$\text{(5)} \qquad s = C_{ss'} \left(C_{s's'} + C_{nn}\right)^{-1} (\ell - Ax)$$

and

$$\text{(6)} \qquad \begin{aligned} s' + n &= (C_{s's'} + C_{nn}) \left(C_{s's'} + C_{nn}\right)^{-1} (\ell - Ax) \\ &= \ell - Ax. \end{aligned}$$

The corresponding derivation for Method Two starts with the model

$$\text{(7)} \qquad \underset{n1}{\ell} = \underset{nu}{A}\,\underset{u1}{x} + \underset{nk}{R}\,\underset{k1}{s} + \underset{n1}{n}$$

Here the desired signal s appears explicity in the model and is related to the observation vector ℓ by the coefficient matrix R with dim (n x k).

It has to be emphasized that even though we use in Model Two the same notation for x, s, and n these quantities may be different from the corresponding variables in Model Two. In fact, the purpose of the following considerations is to clarify under what circumstances the corresponding variables x, s, and n are identical.

A rearrangement of the random quantities s and n leads to

$$\text{(8)} \qquad \underset{n1}{\ell} = \underset{nu}{A}\,\underset{u1}{x} + \underset{n\ n+k}{\begin{bmatrix} R \\ I \end{bmatrix}^{T}} \underset{n+k\ 1}{\begin{bmatrix} s \\ n \end{bmatrix}},$$

and with $B^* = \begin{bmatrix} R \\ I \end{bmatrix}$ and $v^* = \begin{bmatrix} s \\ n \end{bmatrix}$ we obtain a model of the same form as for Method One

$$\ell = Ax + B^{*T} v^*.$$

We minimize $v^{*T} Q^{*-1} v^*$ with

$$Q^* = \begin{bmatrix} C_{ss} & 0 \\ 0 & C_{nn} \end{bmatrix}, \tag{9}$$

and obtain from the least squares solution

$$v^* = Q^* B^{*T} (B^* Q^* B^{*T})^{-1} (\ell - A x)$$

and

$$x = (A^T (B^* Q^* B^{*T})^{-1} A)^{-1} A^T (B^* Q^* B^{*T})^{-1} \ell \tag{10}$$

where with equation (9)

$$Q^* B^{*T} = \begin{bmatrix} C_{ss} R^T \\ C_{nn} \end{bmatrix} \text{ and } B^* Q^* B^{*T} = R C_{ss} R^T + C_{nn}.$$

Finally, the components of v^* are

$$s = C_{ss} R^T (R C_{ss} R^T + C_{nn})^{-1} (\ell - A x) \tag{11}$$

and

$$n = C_{nn} (R C_{ss} R^T + C_{nn})^{-1} (\ell - A x) \tag{12}$$

With equations (1) to (6) we draw the following conclusions for Model One:

--The random signal s' is of the same physical nature as the observations ℓ.

--The desired vector s can be of any dimension, k, and of any kind as long as it is correlated with s'. If there exists no correlation between s and s', which means that there exists no dependence of s on the observations ℓ, the covariance $C_{s's}$ becomes zero and by this also the estimate of the signal s.

--Although solution equation (6) for s' + n is trivial as it shows only the basic model (1), it nevertheless reflects a basic characteristic of all collocation methods, which is the reproduction of the measurement--in our case of the trendfree measurement $\ell - Ax$ -- at the sample points.

--If the elements of the covariance matrices are built from global covariance expressions fulfilling Laplace's equation and following the "law of propagation" such as the covariance expressions derived by C. Tscherning and R. Rapp (1974), the solution equation (5) will converge for limit $n \to \infty$ towards the linear operator equations in physical geodesy, as proved by H. Moritz (1975).

From equations (7) and (8) for Model Two we see that the coefficient matrix R has to be given explicitly.

3. Comparison

A formal investigation of the equations for the two models, (1) to (6), respectively (7) to (12), shows the following differences:

1. The least squares norm for $v^T v$ and $v^{*T} v^*$ is obtained with respect to different variance covariance matrices Q, respectively Q*.

2. The difference in Q and Q* causes the different solution equations for the trendfree observation $s' + n$, equation (6), and for the observation noise, n, equation (12). An equivalent equation for n can be deduced in Model One by splitting equation (6) into an estimation equation for s' and one for n.

3. Due to the same reason, i.e. the difference in Q, also the solutions for the parameters x, equations (4) and (10), and for the random signal s, equations (5) and (11), are different although the formulas show formally a similar structure.

The purpose of both models is the optimal estimation of the parameters x and of the random signal s. Thus, the analysis will be concentrated on the solution equations for x and s. We see that for,

$$C_{ss'} = C_{ss} R^T \tag{13}$$

and for

$$C_{s's'} = R C_{ss} R^T \tag{14}$$

the solution equations (4) and (10) and also (5) and (11) become identical. These two expressions can be derived from one single equation,

$$s' = R s \tag{15}$$

The same relation is immediately obtained comparing model equations (2) and (8). The derivation of equations (13) and (14) from equation (15) is obvious:

Equation (15) is multiplied on both sides with s and the statistical expectation is taken. We obtain:

$$E(s's^T) = R\,E(ss^T) \tag{17}$$

This is a discrete form of the well known WIENER-HOPF equation. In geodesy it is the foundation of the "law of propagation of covariances". Again the integral over the earth takes the place of the statistical expectation. The expectations $E(s's^T)$ and $E(ss^T)$ become the covariances $C_{s's}$ and C_{ss}, and equation (17) becomes equation (13).

In the same way, we take the expectation on the square of equation (15) and obtain,

$$E(s's'^T) = R\,E(ss^T)\,R^T. \tag{18}$$

We replace the expectations by the corresponding covariance expressions and deduce equation (14). The comparison shows that the estimates for the parameters x and the random signal s become identical for both models, if only equation (15) is valid. Therefore our further considerations will be concentrated on this equation. We analyze if there exists a relation (15) with dimension k between two quantities related to the anomalous potential of the earth. In order to provide an easy insight into possible agreements and disagreements the discussion is split into filtering, prediction and collocation.

3.1 Filtering

The purpose of least squares filtering is the separation of the noise n from the signal s in the least squares sense. Thereby we assume that the covariance function of the signal and the covariance function of signal plus noise are given. Since filtering is usually carried out in all observation points, we have $k = n$. Because every signal s corresponds to a certain observation ℓ the coefficient matrix will be the unit matrix, $R = I$. Thus, expression (15) becomes

$$s' = Is$$

or

$$s' = s \tag{19}$$

and identical estimation equations (4) and (10), and also (5) and (11) are effective. For least squares filtering both models lead to identical results.

3.2 Prediction

Least squares prediction means the optimal linear estimation of the random signal s from the observations ℓ at k points which are not identical with the n observation points. In physical geodesy this definition has to be refined. We assume that the observation and prediction points are on the same sphere, σ, or in spherical approximation on the earth's surface or on the geoid.

Following Krarup (1969, p. 16), we denote the last two terms in equation (5) with ξ_{Pi},

$$\xi_{Pi} = (C_{s's'} + C_{nn})^{-1} (\ell - Ax), \quad i = 1, \ldots n. \tag{20}$$

The variable ξ_{Pi}, is only dependent on the n observation points P_i but not on the prediction points. The decision at what and at how many points prediction is carried out depends on $C_{ss'}$. The covariance $C_{ss'}$ is a function of the observation points P_i and of the finite or infinite number k of prediction points Q. The most general form for the covariance vector $C_{ss'}$ is derived when each vector element is an analytical function of the prediction point Q, $Q \epsilon \sigma$,

$$C_{ss'} = C_{Pi}(Q).$$

Then formula (5) becomes with (20)

$$s(Q) = C_{Pi}(Q)\,\xi_{Pi} \tag{21}$$

where $s(Q)$ is a continuous function of the predicted signal, defined on σ. This case expresses the limit $k \to \infty$ where s forms a complete set.

Quite different is the situation for Model Two: The corresponding expression for ξ_{Pi} becomes for Method Two, (from (11))

$$\xi_{PiQj} = (R\,C_{ss}\,R^T + C_{nn})^{-1}\,(\ell - A\,x), \quad j = 1, \ldots k. \quad i = 1, \ldots n. \tag{22}$$

In contrast to equation (20), the expression ξ_{PiQj} is dependent on the observation and prediction points since it contains R and R relates by definition, see equation (7), the observations to the signal. Therefore every choice of k makes necessary a new computation of ξ_{PiQj}.

But even if we accept this restriction for ξ, the prediction cannot be solved with Model Two. For, there does not exist a matrix R transforming a finite set of random signals s into a set of signals s', both quantities related to the anomalous potential of the earth. Only for the limit $k \to \infty$ a solution is theoretically meaningful. For this limit the coefficient matrix, R, degenerates towards Dirac's functions,

$$R \to \delta(\psi_{PiQ}) \tag{23}$$

where ψ_{P_1Q} is the spherical distance between the observation point P_1 and any prediction point Q. The basic relation (15) becomes for this limit

$$\lim_{k \to \infty} s'_{P_1} = \frac{1}{4\pi} \int_{\sigma} \delta(\psi_{P_1Q})\, s(Q)\, d\sigma. \tag{24}$$

We draw the conclusions:

The least squares prediction problem can be efficiently treated by Model One for a finite as well as for an infinite number of random signals. With Model Two the prediction problem cannot be treated. The limit relation for $k \to \infty$ between s' and s is given in formula (24). Considering this expression, Model One can be interpreted to be the limit of Model Two for $k \to \infty$.

3.3 Collocation

We distinguish between filtering, prediction, and collocation to provide an easy insight into the special features of both models, although in the geodetic literature the term least squares collocation usually includes filtering and prediction. In least squares collocation gravity quantities at σ or in Σ -- Σ denotes the space outside σ -- are estimated from given observations which may be at σ or in Σ. Thereby, the collocation quantities s and the observations ℓ may be gravity quantities of a different nature such as the disturbing potential, gravity anomalies, and their first or second derivatives. For an exposition see H. Moritz (1973), T. Krarup (1969), and C. Tscherning (1975).

As in least squares prediction, for the more general collocation method no finite relation (15) can be derived, now considering, as mentioned above, s' and s to be gravity quantities of a different nature. Only for the extreme case where s becomes a continuous function on σ, the relation (15) between s' and s is satisfied by one of the well known operator equations in physical geodesy.

Example: The derivation of gravity anomalies from altimeter data, as described by R. Rapp (1974).

Assuming that the altimeter data are synonymous with geoid undulations, the observation vector ℓ and the random signal vector s' will consist of undulations. The random signal s will be a continuous function $s(Q)$ of the desired gravity anomalies. Then equation (15), i.e. the relation between the undulations s' and the gravity anomalies s is Stokes' formula,

$$s'_{P_i} = \frac{R_e}{4\pi G} \int_{\sigma} St(\psi_{P_i Q})\, s(Q)\, d\sigma \tag{25}$$

with:

R_e .... mean radius of the earth
G .... mean gravity over the earth
$St(\psi)$ ..... Stokes' function for the spherical distance ψ.

This means, in order to fulfill the relation between s' and s the finite k-dimensional vector s becomes a continuous function defined on σ. Also in the spectral domain, in terms of a spherical harmonic expansion, only the limit $k \to \infty$ is valid. The random signal s becomes the vector of the spherical harmonic coefficients of the gravity anomaly expansion. We obtain for equation (15),

$$s'_{P_i} = \frac{R_e}{G} \sum_{n=2}^{\infty} \frac{1}{n-1} \sum_{m=0}^{n} \overline{P}_{nm}(\cos\overline{\varphi}_Q)\,(s^c_{nm} \cos m\lambda_Q + s^s_{nm} \sin m\lambda_Q) \tag{26}$$

with:

$\overline{P}_{nm}(\cos\overline{\varphi}_Q)$ .... associated Legendre polynomial of degree n and order m.

The equivalence of both models can only be achieved for the limit $k \to \infty$, where the finite dimensional matrix, R, becomes one of the well known operator expressions.

For any finite k the two models will give different estimates for the random signal s and also for the parameters x.

Collocation based on Model One is carried out by inserting the continuous function obtained from equation (21) by least squares prediction into an operator equation that provides the desired random signal. Thereby, the continuous signal obtained from equation (21) which is of the same physical nature as the observations is transformed into a finite or infinite number k of signals s.

Example: Let us again return to the problem of gravity anomaly recovery from geoid undulations.

From equation (21), an intermediate signal $\tilde{s}$, i.e. a continuous undulation function defined on σ is obtained. The covariance function $C_{Pi}(Q)$ is in this case the autocovariance function $C^{NN}_{Pi}(Q)$ of the geoid undulations. In order to deduce gravity anomalies the signal $\tilde{s}$ is inserted into the boundary condition of physical geodesy,

$$(27) \qquad s_{Qj} = -G\frac{\partial \tilde{s}_{Qj}}{\partial r} - \frac{2G}{R_{e}}\tilde{s}_{Qj}$$

where s_{Qj} is the vector of the gravity anomalies, $j = 1, \ldots k$. With equation (21) we find

$$(28) \qquad s_{Qj} = -G\frac{\partial (C^{NN}_{Pi}(Q)\,\xi_{Pi})}{\partial r} - \frac{2G}{R_{e}}C^{NN}_{Pi}(Q)\,\xi_{Pi}$$

or

$$s_{Qj} = \left(-\frac{\partial (C^{NN}_{Pi}(Q))}{\partial r} - \frac{2}{R_{e}}C^{NN}_{Pi}(Q)\right)G\,\xi_{Pi}\,.$$

We see that in equation (28) the operator equation (27) is only applied on the covariance $C^{NN}_{PI}(Q)$ leaving ξ_{PI} untouched. In other words, the "law of propagation of covariances" is applied on $C^{NN}_{PI}(Q)$. The expression in parenthesis becomes the crosscovariance between the geoid undulations and gravity anomalies, $C^{N\Delta g}_{PI}(Q)$. We derive from equation (28),

$$s_{QJ} = C^{N\Delta g}_{PIQJ}\, G\xi_{PI} . \tag{29}$$

In Model One the only approximation is performed in the prediction step, equation (21), where from n observations a function covering the entire sphere σ is estimated. The second step, where this function is inserted into an operator equation, is carried out without any approximation.

In Model Two a coefficient matrix R has to be established. In all practical situations the dimension k will be chosen to be finite as shown for example in (Schwarz, 1974). In order to solve a particular problem, an integral operator or its development into an infinite Legendre polynomial connecting the signal with the observations has to be approximated by a (n x k)-dimensional finite matrix R.

Example: For the application of Model Two to our problem, the recovery of gravity anomalies from altimeter data, a finite approximation to equations (25) and (26) has to be formed. In one case the signal vector s will consist of gravity anomalies and the coefficient matrix R of n x k values of Stokes' function. In the other case, the coefficients of a spherical harmonic expansion of the gravity anomaly field are the elements of the vector s and R consists of n x k spherical harmonics.

In contrast to Model One, for Model Two the operator equations have to be approximated by a finite system.

Summarizing the situation in least squares collocation, we see: The equivalence between Model One and Model Two can only be achieved for the global limit $k \to \infty$. Then the basic expression (15) becomes one of the operator equations of physical geodesy. For any finite k collocation by Model One and by Model Two will result in different estimates for s and for x. In Model One, the only approximation is introduced for the derivation of a function of the type of the observed quantities covering the entire sphere σ, whereas in Model Two, one of the integral or spectral operators is approximated by a finite system.

Part II: Numerical Example

In this second part a numerical example is presented that allows us to discuss some details in the comparison of the two least squares estimation methods. The example treats the determination of spherical harmonic coefficients from geoid heights. The solution equations of the two methods are unfortunately very similar for this example because of the orthogonality properties of the spherical harmonics, as will be explained later in the paper. But on the other hand this example has the advantage that it can be judged independently of the chosen data. In addition, it is to a certain extent complementary to (Schwarz, 1975) where for a similar problem regular least squares adjustment using no apriori covariance information is compared with method (2).

We assume to have given 60 geoid heights free of errors in a global spherical grid from latitude $\varphi = 60°$ to $\varphi = -60°$ and from longitude $\lambda = 0°$ to $\lambda = 330°$ in 30° intervals. The grid allows the resolution of spherical harmonic coefficients up to about degree $\ell = 6$ and order $|m| = 5$. Although it would theoretically be possible to determine from both methods, method (1) as well as method (2), an infinite number of coefficients -- where the coefficients with $\ell > 6$ would not have any meaning besides that they would contribute to reproduce the original geoid heights -- we solve only for the coefficients up to $\ell_0 = 4$ (ℓ_0 ... maximum degree) and $|m| = 4$. That means that we try to determine $k = 21$ coefficients (neglecting degree zero and one) given $n = 60$ observations.

The spherical harmonic expansion of the vector, $\underline{u}$, of 60 geoid heights is,

$$(30) \qquad \underset{n\,1}{\underline{u}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \bar{a}_{\ell m} \bar{Y}_{\ell m} (P_i) = \underset{n\,\infty}{\underline{\underline{\Gamma}}} \; \underset{\infty\,1}{\underline{a}} , \quad i = 1, \ldots, 60;$$

with $\bar{a}_{\ell m}$ ..... fully normalized spherical harmonic coefficients of degree ℓ and order m of the geoid height

$\underline{a}$ ...... vector of these coefficients (dim. $\infty \cdot 1$)

$\overline{Y}_{\ell m}$..... fully normalized spherical harmonics of degree ℓ and order m

Γ...... matrix to the harmonics (dim. $n \cdot \infty$)

and with

$$\overline{Y}_{\ell m}(P_i) = (-1)^m \left[\frac{(2-\delta_{om})(2\ell+1)(\ell-m)!}{(\ell+m)!} \right]^{\frac{1}{2}} P_{\ell m}(\sin\varphi_{i})\, e^{im\lambda_i}$$

where δ_{om}..... Kronecker symbol

$P_{\ell m}(\sin\varphi_i)$. associated Legendre functions.

For later needs the spherical harmonic development is splitted into a k-dimentional part one and a $(\infty - k + 1)$ - dimensional part two,

$$(31) \qquad \underset{n\,1}{\underline{u}} = \underset{n\,\infty}{\underline{\underline{\Gamma}}}\,\underset{\infty\,1}{\underline{a}} = \underset{n\,k}{\underline{\underline{R}}}\,\underset{k\,1}{\underline{a}_1} + \underset{n\,(\infty-k+1)}{\underline{\underline{S}}}\;\underset{(\infty-k+1)\,1}{\underline{a}_2}$$

<u>Method (1):</u>

Based on the given 60 geoid heights an approximation to the "true" global geoid height function u(Q) is derived from

$$(32) \qquad \hat{u}(Q) = \underline{C}_{uu}(Q, P_j)\, \underline{\underline{C}}_{uu}(P_j, P_i)^{-1}\, \underline{u}(P_i)$$

where $\underline{C}_{uu}$ is the autocovariance matrix of the geoid heights. The estimates for the geoid height spherical harmonic coefficients are obtained by inserting expression (32) into an integral formula,

$$\hat{a}_{\ell m} = \frac{1}{4\pi} \int_\sigma \hat{u}(Q)\, \overline{Y}_{\ell m}(Q)\, d\sigma$$

$$= \left[\frac{1}{4\pi} \int_\sigma \underline{C}_{uu}(Q, P_j)\, \overline{Y}_{\ell m}(Q)\, d\sigma\right] \underline{\underline{C}}_{uu}(P_j, P_i)^{-1}\, \underline{u}(P_i)$$

By using for the covariance function the expression

$$C_{uu}(Q, P) = \sum_{\ell=0}^{\infty} d_\ell\; s^{\ell+2}\; P_\ell(\cos\psi_{QP})$$

where $d_\ell = \frac{R^2}{G^2} \frac{c_\ell}{(\ell-1)^2}$ ... geoid height degree variance

$s = \left(\frac{R_e}{R_{Bje}}\right)^2$ .... square ratio of the mean earth radius R_e to a Bjerhammar sphere with $R_{Bje} < R_e$

we obtain for $\hat{a}_{\ell m}$,

(33) $$\hat{a}_{\ell m} = \underline{\left(\frac{d_\ell}{2\ell+1}\; s^{\ell+1}\; \overline{Y}_{\ell m}(P_j)\right)}\; \underline{\underline{C}}_{uu}(P_j, P_i)^{-1}\; \underline{u}(P_i)$$

and denoting

(34) $$\underline{C}_{au} = \underline{\left(\frac{d_\ell}{2\ell+1}\; s^{\ell+1}\; \overline{Y}_{\ell m}(P_j)\right)}$$

we find the estimation formula for the geoid height coefficients

$$(35)\qquad \underset{1\,1}{\hat{a}_{\ell m}} = \underset{1\,n}{\underline{C}_{au}}\ \underset{n\,n}{\underline{\underline{C}}_{uu}^{-1}}\ \underset{n\,1}{\underline{u}}$$

Because of the global character of the approximation $\hat{u}(Q)$ any geoid height coefficient may be estimated by equation (35) independently from one another. Since we solve only for the lowest $k = 21$ coefficients (zero and first degree excluded) we use the notation of equation (31) and rewrite equation (35) to

$$(36)\qquad \underset{k\,1}{\hat{\underline{a}}_1} = \underset{k\,n}{\underline{\underline{C}}_{au}}\ \underset{n\,n}{\underline{\underline{C}}_{uu}^{-1}}\ \underset{n\,1}{\underline{u}}$$

and

$$(37)\qquad \underset{k\,1}{\hat{\underline{a}}_1} = \underset{k\,n}{\underline{\underline{C}}_{au}}\ \underset{n\,n}{\underline{\underline{C}}_{uu}^{-1}}\ \underset{n\,k}{\underline{\underline{R}}}\ \underset{k\,1}{\underline{a}_1} + \underset{k\,n}{\underline{\underline{C}}_{au}}\ \underset{n\,n}{\underline{\underline{C}}_{uu}^{-1}}\ \underset{n\,(\infty-k+1)}{\underline{\underline{S}}}\ \underset{(\infty-k+1)\,1}{\underline{a}_2}$$

Denoting

$$\underline{\underline{A}}_1 = \underline{\underline{C}}_{au}\ \underline{\underline{C}}_{uu}^{-1}\ \underline{\underline{R}}$$

and

$$\underline{\underline{A}}_2 = \underline{\underline{C}}_{au}\ \underline{\underline{C}}_{uu}^{-1}\ \underline{\underline{S}}$$

we obtain

$$(38)\qquad \hat{\underline{a}}_1 = \underline{\underline{A}}_1\,\underline{a}_1 + \underline{\underline{A}}_2\,\underline{a}_2$$

Matrix $\underline{\underline{A}}_1$ relates the unknown low frequency spherical harmonic coefficients of the given geoid heights to the unknown spherical harmonic coefficients to be estimated. In an ideal situation $\underline{\underline{A}}_1$ is expected to be the identity matrix $\underline{\underline{I}}$. Matrix

$\underline{\underline{A}}_2$ relates the coefficients with $\ell > \ell_0$ to the coefficients to be estimated. Ideally $\underline{\underline{A}}_2$ should be the zero matrix $\underline{\underline{O}}$. In a practical situation like ours where a finite but regular grid with observations defines a smallest resolvable wavelength (or Nyquist frequency) the matrix $\underline{\underline{A}}_2 \neq \underline{\underline{O}}$ generates the alaising effect from the coefficients with $\ell > \ell_0$.

Method (2)

Since we solve in our case only for k = 21 spherical harmonic geoid height coefficients, method (2) is based on a finite approximation to the infinite series expansion of equation (30). We express the geoid heights by a finite series of spherical harmonics accepting a certain model error, ϵ,

(39) $$\underset{n\,1}{\underline{u}} = \underset{n\,k}{\underline{\underline{R}}}\ \underset{k\,1}{\underline{a}_1} + \underset{n\,1}{\epsilon}$$

The solution equation for the unknown parameter vector $\underline{a}_1$ is, using equation (11)

(40) $$\underset{k\,1}{\hat{\underline{a}}_1} = \underset{k\,k}{\underline{\underline{C}}_{aa}}\ \underset{k\,n}{\underline{\underline{R}}^T} (\underset{n\,k}{\underline{\underline{R}}}\ \underset{k\,k}{\underline{\underline{C}}_{aa}}\ \underset{k\,n}{\underline{\underline{R}}^T} + \underset{n\,n}{\underline{\underline{C}}_{nn}})^{-1}\ \underset{n\,1}{\underline{u}}$$

Here $\underline{\underline{C}}_{aa}$ is the autocovariance matrix of the spherical harmonic coefficients. Because of the orthogonality of the spherical harmonics it becomes a diagonal matrix with elements $s^{\ell+1} \dfrac{d_\ell}{2\ell + 1}$, and it holds the relationship

(41) $$\underset{k\,n}{\underline{\underline{C}}_{au}} = \underset{k\,k}{\underline{\underline{C}}_{aa}}\ \underset{k\,n}{\underline{\underline{R}}^T}.$$

Relation (41) based on the orthogonality properties of spherical harmonics and corresponding to equation (13) of part 'one' is the reason that for this specific example, i.e. the determination of spherical harmonic coefficients from a gravity function, the estimation equations of method (1) and method (2) become very similar.

In addition, the matrix product $\underline{\underline{R}}\ \underline{\underline{C}}_{aa}\ \underline{\underline{R}}^T$ is the autocovariance matrix for geoid height including wavelengths up to ℓ_0. We may write

$$\text{(42)} \qquad \underline{\underline{C}}^{\ell_0}_{2,uu} = \underline{\underline{R}}\ \underline{\underline{C}}_{aa}\ \underline{\underline{R}}^T$$

and

$$\text{(43)} \qquad \underline{\underline{C}}_{uu} = \underline{\underline{C}}^{\ell_0}_{2,uu} + \underline{\underline{C}}^{\infty}_{\ell_0+1,uu}$$

From equations (42) and (43) follows that relation (14) of part 'one' does not hold for our example which means that the corresponding estimation equations for both methods are not identical.

We obtain for equation (40) using (41) and (42)

$$\text{(44)} \qquad \hat{\underline{a}}_1 = \underline{\underline{C}}_{au}\ (\underline{\underline{C}}^{\ell_0}_{2,uu} + \underline{\underline{C}}_{nn})^{-1}\,\underline{u}$$

and with (31)

$$\text{(45)} \qquad \hat{\underline{a}}_1 = \underline{\underline{C}}_{au}\ (\underline{\underline{C}}^{\ell_0}_{2,uu} + \underline{\underline{C}}_{nn})^{-1}\,\underline{\underline{R}}\ \underline{a}_1 + \underline{\underline{C}}_{au}\ (\underline{\underline{C}}^{\ell_0}_{2,uu} + \underline{\underline{C}}_{nn})^{-1}\,\underline{\underline{S}}\ \underline{a}_2 .$$

Denoting

$$\underline{\underline{B}}_1 = \underline{\underline{C}}_{au}\ (\underline{\underline{C}}^{\ell_0}_{a,uu} + \underline{\underline{C}}_{nn})^{-1}\,\underline{\underline{R}} ,$$

and

$$\underline{\underline{B}}_2 = \underline{\underline{C}}_{au} (\underline{\underline{C}}^{\ell_0}_{a,uu} + \underline{\underline{C}}_{nn})^{-1} \underline{\underline{S}}$$

the corresponding equation to (38) becomes

$$\hat{\underline{a}} = \underline{\underline{B}}_1 \underline{a}_1 + \underline{\underline{B}}_2 \underline{a}_2 \tag{46}$$

where again in an ideal situation $\underline{\underline{B}}_1$ should be the identity matrix and $\underline{\underline{B}}_2$ the zero matrix.

Although the given geoid heights are assumed to be free of errors the matrix $\underline{\underline{C}}_{nn}$ is included in the estimation equation of method (2). It may be used (i) to model partially the model error, ϵ, in equation (39) and (ii) to stabilize the numerical solution of equations (44) to (46). In order to see the influence of the noise matrix $\underline{\underline{C}}_{nn}$ on the estimates two different models were used: model 2ℓ ("ℓ" for low noise level) with $\underline{\underline{C}}_{nn} = 0.01\,\underline{\underline{I}}$ and model 2h ("h" for high noise level) with $\underline{\underline{C}}_{nn} = \sigma^2 \underline{\underline{I}}$ with

$$\sigma^2 = \sum_{\ell=\ell_0+1}^{\infty} s^{\ell+1} d_\ell$$

Model 2h is corresponding with ϵ to be interpreted as white noise with variance equal to the variance of the short wavelengths ($\ell > \ell_0$) not contained in $\underline{\underline{C}}^{\ell_0}_{a,uu}$.

Numerical Results and Their Interpretation

First, the matrices $\underline{\underline{A}}_1$ for method (1) and $\underline{\underline{B}}_1$ for method (2) have been computed. They depend on the spherical harmonic coefficients to be estimated and on the location of the "observations" but not on their magnitude. Therefore they give an objective picture of the transformation from $\underline{a}_1$ to $\hat{\underline{a}}_1$. For the described data grid the results for matrix $\underline{\underline{A}}_1$ and for matrix $\underline{\underline{B}}_1$ for version 2ℓ (low noise) and 2h (high noise) are given in tables one to three.

The result for matrix $\underline{\underline{B}}_1$ of model 2ℓ is within 1% a perfect identity matrix as shown in Table 2. Neglecting the influence of the second right hand term of equation (46) this means that method (2) would produce perfect coefficient estimates using model 2ℓ. Matrix $\underline{\underline{B}}_1$ (table three) using model 2h shows some small off-diagonal terms and the diagonal terms slightly damped with increasing degree and order.

In order to understand the effect of $\underline{\underline{A}}_2$ and $\underline{\underline{B}}_2$ as well as of the not ideal matrix $\underline{\underline{A}}_1$ or $\underline{\underline{B}}_1$ on the estimated coefficients we generated geoid height spherical harmonic coefficients from the GEM 8 set of potential coefficients (Wagner et als., 1976) which is complete up to degree 25 with some additional terms, and estimated the vector $\hat{\underline{a}}_1$ up to $\ell_0 = 4$ and $|m| = 4$ using equations (38) and (46). We split the estimated coefficients into three parts:

(1) The part produced by the diagonal terms of $\underline{\underline{A}}_1$ and $\underline{\underline{B}}_1$ (all of them should be one in an ideal situation): The results are listed for all three models in column 2 of tables four to six.

Because of the perfect structure of $\underline{\underline{B}}_1$ for model 2ℓ the estimates in column 2 are equal to the input coefficients. For method (1) and for model 2h the coefficients are reproduced with small distortions because of the damped terms on the diagonal of $\underline{\underline{A}}_1$ and $\underline{\underline{B}}_1$ respectively.

(2) The part produced by the off-diagonal terms of $\underline{\underline{A}}_1$ and $\underline{\underline{B}}_1$ (all of them should be zero): The results are given in column 3 of tables four to six.

Also in column 3 there are only very small distortions always less than 1/1000 of the corresponding coefficient for model 2ℓ. The distrubing contribution caused by the off-diagonal terms of $\underline{\underline{A}}_1$ and $\underline{\underline{B}}_1$ is however significant for method (1) and for model 2h.

(3) The part produced by the matrices $\underline{\underline{A}}_2$ and $\underline{\underline{B}}_2$ which is the alaising effect of the coefficients with $\ell > \ell_0$ on $\hat{\underline{a}}_1$ (all elements should be zero in an ideal situation). The results are given in column 4 of tables four to six.

The dimension of the matrices $\underline{\underline{A}}_2$ and $\underline{\underline{B}}_2$ is only $n \cdot (673 - k + 1)$ instead of $n \cdot (\infty - \bar{k} + 1)$. The number 673 is thereby the full number of potential coefficients up to $\ell = 25$ and $|m| = 25$ excluding degree zero and one.

We see in column 4 that the alaising effect for method (2) model 2ℓ is higher as compared to method (1) and also to method (2) model 2h. But since the magnitude of the input coefficients decreases rapidly with increasing degree, e.g. expressed by Kaula's rule of thumb, the disturbing influence on the estimated coefficients is comparably small.

Column 5 of the tables four to six contains the sum of all three parts which is the vector of the estimated coefficients, $\hat{\underline{a}}_1$ and in column 6 the vector, $\underline{a}_1$, of the input coefficients is listed. The "best" estimates for the spherical harmonic coefficients in the sense that the estimated coefficients are closest to the input coefficients is achieved by method (2) with the low noise model 2ℓ. The reasons for this fact shall be more closely analyzed in the sequel.

For convenience we denote

$$\underline{\underline{C}}_{2,uu}^{\ell_0} = \underline{\underline{\bar{C}}}$$

and

$$\underline{\underline{C}}_{\ell_0+1,uu}^{\infty} = \underline{\underline{\Delta C}}$$

	C 2 0	C 2 1	S 2 1	C 2 2	S 2 2	C 3 0	C 3 1	S 3 1	C 3 2	S 3 2	C 3 3	S 3 3	C 4 0	C 4 1	S 4 1	C 4 2	S 4 2	C 4 3	S 4 3	C 4 4	S 4 4
C 2 0	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	-11.	0.	0.	0.	0.	0.	0.	0.	0.
C 2 1	0.	99.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.	0.	0.	0.
S 2 1	0.	0.	99.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.	0.	0.
C 2 2	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.	0.
S 2 2	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.
C 3 0	0.	0.	0.	0.	0.	95.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 1	0.	0.	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 1	0.	0.	0.	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 2	0.	0.	0.	0.	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 4 0	-2.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	69.	0.	0.	0.	0.	0.	0.	0.	0.
C 4 1	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.	0.	0.	0.	0.
S 4 1	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.	0.	0.	0.
C 4 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.	0.	0.
S 4 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.	0.
C 4 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.
S 4 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.
C 4 4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	87.	0.
S 4 4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	87.

<u>Table 1.</u> Matrix $\underline{\underline{A}}_1$ · 001 for method (1).

	C 2 0	C 2 1	S 2 1	C 2 2	S 2 2	C 3 0	C 3 1	S 3 1	C 3 2	S 3 2	C 3 3	S 3 3	C 4 0	C 4 1	S 4 1	C 4 2	S 4 2	C 4 3	S 4 3	C 4 4	S 4 4
C 2 0	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 2 1	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 2 1	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 2 2	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 2 2	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 0	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 1	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 1	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 2	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 4 0	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.	0.
C 4 1	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.	0.
S 4 1	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.	0.
C 4 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.	0.
S 4 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.	0.
C 4 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.	0.
S 4 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.	0.
C 4 4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.	0.
S 4 4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	100.

<u>Table 2.</u> Matrix $\underline{B}_1 \cdot 0.01$ for method (2) model 2^{ℓ}.

	C20	C21	S21	C22	S22	C30	C31	S31	C32	S32	C33	S33	C40	C41	S41	C42	S42	C43	S43	C44	S44
C 2 0	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	-2.	0.	0.	0.	0.	0.	0.	0.	0.
C 2 1	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	3.	0.	0.	0.	0.	0.	0.	0.
S 2 1	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	3.	0.	0.	0.	0.	0.	0.
C 2 2	0.	0.	0.	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.	0.
S 2 2	0.	0.	0.	0.	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	2.	0.	0.	0.	0.
C 3 0	0.	0.	0.	0.	0.	96.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 1	0.	0.	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 1	0.	0.	0.	0.	0.	0.	0.	98.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 2	0.	0.	0.	0.	0.	0.	0.	0.	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	97.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 3 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	96.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
S 3 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	96.	0.	0.	0.	0.	0.	0.	0.	0.	0.
C 4 0	-0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	81.	0.	0.	0.	0.	0.	0.	0.	0.
C 4 1	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	91.	0.	0.	0.	0.	0.	0.	0.
S 4 1	0.	0.	1.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	91.	0.	0.	0.	0.	0.	0.
C 4 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.	0.	0.
S 4 2	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	90.	0.	0.	0.	0.
C 4 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	88.	0.	0.	0.
S 4 3	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	88.	0.	0.
C 4 4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	87.	0.
S 4 4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	87.

Table 3. Matrix $\underline{B}_1 \cdot 0.01$ for method (3) model 2h.

		from diagonal of $\underline{A}_1$	from off-diagonal of $\underline{A}_1$	from $\underline{B}_1$	estimate sum 2, 3, 4	input
1		2	3	4	5	6
2	0	0.06517	0.16831	0.07428	0.30776	0.06713
2	1	-0.00041	-0.08011	-0.06661	-0.14714	-0.00042
2	1	0.00180	-0.07056	0.03897	-0.02979	0.00183
2	2	15.26656	0.04255	0.01277	15.32188	15.54655
2	2	-8.75019	0.08158	-0.02123	-8.68984	-8.91066
3	0	5.79696	0.00009	-0.01380	5.78325	6.11637
3	1	12.70010	-0.00003	0.09497	12.79504	12.96395
3	1	1.56016	0.00005	-0.05432	1.50589	1.59258
3	2	5.60725	-0.00007	-0.02939	5.57779	5.73012
3	2	-3.89300	-0.00003	0.05178	-3.84126	-3.97831
3	3	4.44619	0.00001	0.00032	4.44652	4.57315
3	3	8.79654	0.00001	-0.01617	8.78038	9.04774
4	0	-1.10642	-0.00139	-0.00610	-1.11391	-1.59498
4	1	-3.10289	0.00004	-0.07775	-3.18060	-3.42616
4	1	-2.73573	0.00003	0.04540	-2.69031	-3.02075
4	2	2.00418	0.05235	0.04531	2.10184	2.21423
4	2	3.84163	-0.02995	-0.07760	3.73408	4.24428
4	3	5.69139	-0.00004	0.00032	5.69168	6.28301
4	3	-1.13613	0.00004	0.04370	-1.09238	-1.25423
4	4	-1.09034	-0.00004	-0.00500	-1.09539	-1.24628
4	4	1.70339	0.0	-0.00890	1.69448	1.94700

Table 4. Estimated Geoid Height Spherical Harmonic Coefficients for Method (1).

		from diagonal of $\underline{\underline{A}}_2$	from off-diagonal of $\underline{\underline{A}}_2$	from $\underline{\underline{B}}_2$	estimate sum of 2, 3, & 4	input
1		2	3	4	5	6
2	0	0.06713	0.00062	0.13652	0.20427	0.06713
2	1	-0.00042	0.00046	-0.02631	-0.02627	-0.00042
2	1	0.00183	0.00048	0.07752	0.07982	0.00183
2	2	15.54655	-0.00037	0.05181	15.59799	15.54655
2	2	-8.91066	-0.00069	0.01823	-8.89312	-8.91066
3	0	6.11637	0.00007	-0.03422	6.08221	6.11637
3	1	12.96395	-0.00003	0.14708	13.11100	12.96395
3	1	1.59258	0.00004	-0.02259	1.57003	1.59258
3	2	5.73012	-0.00007	-0.00068	5.72937	5.73012
3	2	-3.97831	-0.00004	0.10914	-3.86921	-3.97831
3	3	4.57315	0.00001	0.03952	4.61268	4.57315
3	3	9.04774	0.00001	0.01678	9.06453	9.04774
4	0	-1.59498	-0.00015	0.06859	-1.52654	-1.59498
4	1	-3.42616	0.00005	-0.04637	-3.47249	-3.42616
4	1	-3.02075	0.00002	0.08910	-2.93163	-3.02075
4	2	2.21423	-0.00447	0.09618	2.30595	2.21423
4	2	4.24428	0.00262	-0.05863	4.18827	4.24428
4	3	6.28301	-0.00005	0.03952	6.32248	6.28301
4	3	-1.25423	0.00005	0.09197	-1.16222	-1.25423
4	4	-1.24628	-0.00005	0.03340	-1.21293	-1.24628
4	4	1.94700	0.0	0.02892	1.97592	1.94700

Table 5. Estimated Geoid Height Spherical Harmonic Coefficients for Method (2) Model 2ℓ.

		from diagonal of $\underline{\underline{A}}_s$	from off-diagonal of $\underline{\underline{A}}_s$	from $\underline{\underline{B}}_s$	estimate sum of 2, 3, & 4	input
1		2	3	4	5	6
2	0	0.06453	0.04917	0.09261	0.20631	0.06713
2	1	-0.00040	-0.14880	-0.06678	-0.21599	-0.00042
2	1	0.00176	-0.13113	0.03895	-0.09042	0.00183
2	2	14.79840	0.05210	0.01329	14.86379	15.54655
2	2	-8.48185	0.09988	-0.02234	-8.40431	-8.91066
3	0	5.72954	0.00006	-0.06883	5.66078	6.11637
3	1	12.41922	-0.00003	0.10329	12.52248	12.96395
3	1	1.52566	0.00004	-0.05925	1.46645	1.59258
3	2	5.42104	-0.00007	-0.03778	5.38319	5.73012
3	2	-3.76372	-0.00003	0.06611	-3.69764	-3.97831
3	3	4.28032	0.00001	0.00025	4.28058	4.57315
3	3	8.46838	0.00001	-0.02103	8.44736	9.04774
4	0	-1.12958	-0.00049	0.02028	-1.10979	-1.59498
4	1	-2.92038	0.00004	-0.07350	-2.99384	-3.42616
4	1	-2.57481	0.00003	0.04282	-2.53196	-3.02075
4	2	1.83262	0.06530	0.04720	1.94512	2.21423
4	2	3.51279	-0.03738	-0.08108	3.39433	4.24428
4	3	5.05440	-0.00004	0.00025	5.05461	6.28301
4	3	-1.00897	0.00004	0.04244	-0.96649	-1.25423
4	4	-0.97537	-0.00004	-0.00454	-0.97995	-1.24628
4	4	1.52378	0.0	-0.00804	1.51574	1.94700

Table 6. Estimated Geoid Height Spherical Harmonic Coefficients for Method (2) Model 2h.

where "Δ" indicates that the elements of $\underline{\underline{\Delta C}}$ are usually smaller than these of $\underline{\underline{\bar{C}}}$, especially when the maximum degree ℓ_0 is high. Under the usual conditions of convergency we expand $\underline{\underline{C}}_{uu}^{-1} = (\underline{\underline{\bar{C}}} + \underline{\underline{\Delta C}})^{-1}$ into a power series,

$$\underline{\underline{C}}_{uu}^{-1} = (\underline{\underline{\bar{C}}} + \underline{\underline{\Delta C}})^{-1} = \underline{\underline{\bar{C}}}^{-1} - \underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1} + \underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1} - \dots \tag{47}$$

Denoting the geoid height part generated by the spherical harmonic expansion up to ℓ_0 with u_1 and the residual geoid height with u_2, equation (31) becomes

$$\underline{u} = \underline{\underline{R}}\,\underline{a}_1 + \underline{\underline{S}}\,\underline{a}_2 = \underline{u}_1 + \underline{u}_2 \tag{48}$$

Equation (37) becomes with (47) and (48)

$$\hat{\underline{a}}_1 = \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{u}_1 + \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{u}_2 - \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{u}_1 - \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{u}_2 + \dots \tag{49}$$

or

$$\hat{\underline{a}}_1 = \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{R}}\,\underline{a}_1 + \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{S}}\,\underline{a}_2 - \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{R}}\,\underline{a}_1 - \underline{\underline{C}}_{au}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{S}}\,\underline{a}_2 + \dots \tag{50}$$

Multiplying equation (50) on both sides with R and using equations (41) and (42) we obtain

$$\underline{\underline{R}}\,\hat{\underline{a}}_1 = \underline{\underline{R}}\,\underline{a}_1 + \underline{\underline{S}}\,\underline{a}_2 - \underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{R}}\,\underline{a}_1 - \underline{\underline{\Delta C}}\,\underline{\underline{\bar{C}}}^{-1}\underline{\underline{S}}\,\underline{a}_2 + \dots. \tag{51}$$

With this series expansion it is possible to discuss equations (49) to (51) term by term.

case 1: Truncation of series (49) after the first right hand term: Then the equation represents an "exact" collocation solution in a k-dimensional space, i.e. in a situation where the anomalous potential of the earth can be represented by a series of spherical harmonics up to degree ℓ_0 with k terms. Equation (51) shows that truncation after the first right hand term results in perfect reproduction of $\underset{\sim}{\hat{a}_1}$.

<u>case 2</u>: Truncation of series (49) to (51) after the first two terms: The first two right hand terms of (50) are identical to these of equation (46) for method (2) if $\underline{\underline{C}}_{nn} = \underline{\underline{O}}$ which is approximately the case for model 2ℓ.

That means that the estimation equations for model 2ℓ with only the first right hand term reflect the situation described in case 1, as shown in table 2 where

$$\underline{\underline{C}}_{au}\ \underline{\underline{\bar{C}}}^{-1}\ \underline{\underline{R}} = \underline{\underline{B}}_1$$

is a perfect identity matrix. The second term expresses the alaising effect. As is obvious from equations (50) and (51) the magnitude of the elements of $\underline{a}_2$ directly influences the disturbing alaising effect. Therefore since the magnitude of the coefficients of quantities related to the earth's anomalous potential decreases in the mean with increasing frequency, the alaising effect is comparably small as long as low harmonic coefficients are estimated.

<u>case 3</u>: The complete formulas (49) to (51) represent the collocation solution for method (1). The first and higher order terms in (49) containing $\underline{u}_1$ and in (50) containing $\underline{a}_1$ cause the damping of the diagonal terms and the appearance of off-diagonal terms in matrix $\underline{\underline{A}}_1$ (table 1). On the other hand, the first and higher order terms containing $\underline{u}_2$ and a_2 respectively reduce significantly the alaising effect as can be seen by comparing columns 4 of tables 4 and 5.

The matrix product $\underline{\underline{\Delta C}}\ \underline{\underline{\bar{C}}}^{-1}$ which occurs in all first and higher order terms of equation (49) to (51) would become zero if instead of a finite point grid a global geoid height function would be given, for then $\underline{\underline{\Delta C}}$ and $\bar{\underline{C}}^{-1}$ would be orthogonal. For this case the solutions for method (1) and method (2) would become identical as already shown in part one. For all finite point grids this orthogonality relationship is lost and causes the differences in the solutions for both methods.

Using method (1) the approximation to a gravity function is carried out in an infinite dimensional Hilbert space with kernel function C represented by an infinite dimensional Legendre series. Therefore it would be advisable to solve for all coefficients up to infinity. Only then a perfect reproduction of the given geoid height from the estimated coefficients is ensured.

Finally, we see that method (2) using model 2h yields almost the same results as method (1). Small differences may be explained by the fact that for method (1) $\underline{C}_{uu}$ is used which contains all wavelength up to degree and order infinity, whereas for method (2) model 2h in equation (43) the second right had term $\underline{C}^{\infty}_{\ell_0+1,uu}$ is approximated by

$$\underline{\underline{\Delta C}} \doteq \underline{C}_{nn} = \sigma^2 \underline{I} \quad \text{with } \sigma^2 = \sum_{\ell = \ell_0+1}^{\infty} s^{\ell+1} d_\ell$$

Because of the finite point grid the coefficients with $\ell > \ell_0$ cannot be resolved. That means that the low wavelength part of the geoid heights basically behaves like white noise or in other words $\underline{\underline{\Delta C}}$ becomes approximately $\underline{C}_{nn}$.

From column 5 of tables 4 to 5 we see that all three models produce a reasonable recovery of the input geoid height spherical harmonic coefficients. Although method (2) with model 2ℓ produces the most satisfactory results the underlying model can hardly be justified since the low wavelengths components of the given geoid heights are not properly modeled.

Part III: Conclusions

The two least squares models, described by the equtions (1), (2), (7), and (8) can be distinguished by their different variance-covariance matrices Q and Q^*, compare equations (3) and (9). The differences in Q and Q^* lead to different formulas for the estimation of the parameters x, equations (4) and (10), and for the random signal s, equations (5) and (11). Only if the relation

$$s' = Rs \tag{15}$$

between s and s' is valid, the estimation equations for x and s become identical. In physical geodesy where s' and s are quantities related to the earth anomalous potential, equation (15) is only fulfilled by one of the well known operator equations, such as the Poisson, Stokes, or Vening-Meinesz equations. In the frequency domain, where R expresses a spherical harmonic development s becomes the infinite dimensional vector of spherical harmonic coefficients of this development. In the parameter domain, equation (15) will be one of the integral equations of physical geodesy and we have to assume s to be given continuously at the entire sphere, σ.

For any finite dimension of the random signal vector s we obtain different results for x and s from Model One as compared to Model Two. The only approximation necessary to solve the finite problem is performed in Model One for the derivation of a continuous function interpolated from the observations. After inserting this interpolated function into the operator equation suited for the derivation of the desired gravity quantity, the operator equation is solved without any approximation, whereas for Model Two the operator equation itself is approximated by a finite matrix system. Within this respect, Model One is superior to Model Two. The advantage of Model Two lies in the fact, that -- as already mentioned in the introduction -- the solution of a large linear system with dimension equal to the number of observations can be replaced by the solution of a linear system with dimension equal to the number of unknowns, since R appears explicitly in the solution equations (10), (11) and allows the application of certain matrix manipulations, compare for example (U. Uotila, unpublished paper, (1973) or P. B. Liebelt (1967) pp. 27).

Finally, any problem which can be expressed by Model Two, i.e. for which an explicit form of the coefficient matrix R can be established, can also be solved by Model One. This is obvious since with C_{ss} -- assumed to be given in both models--and R the covariances $C_{ss'}$ and $C_{s's'}$ necessary in Model One can be derived from equations (13) and (14). The estimated parameter vector x and random signal vector s will be identical.

The estimated value of a certain gravity parameter depends not only on the special features of the two models but to a high extent on the characteristics of the gravity field as well. As an example low degree and order ($\ell = |m| = 4$) spherical harmonic coefficients of the geoid height have been estimated from 60 geoid heights which have been generated from the GEM-8 set of potential coefficients ($\ell = |m| = 25$) and were given in a global regular grid over the sphere. For this problem the estimation equations become almost identical for both methods with the only difference that for Method One the covariances between the observations contain all wavelengths inherent also in the spectrum of the observations whereas these of Method Two contain only the wavelength range of the coefficients to be estimated. Because of this a considerable alaising effect would be expected for Method Two. But the results show that because of the rapid decrease of the degree variances of geoid heights with increasing degree the disturbing effect of the shorter wavelengths ($4 < \ell \leq 25$) is almost negligible. The best recovery of the harmonic coefficients was achieved using Model Two and assuming that the given geoid heights contain no wavelengths with $\ell > 4$. Method Two modeling the wavelengths between $\ell = 5$ and $\ell = 25$ as white noise yielded about the same results as Method One.

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